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LETTER TO THE EDITOR

Kinetic representation of discrete integrable systems

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Abstract. The basic variables in discrete Lax equations can be represented as moments of the one-particle distribution function satisfying certain Vlasov-type kinetic dynamics. These kinetic equations are Hamiltonian, and the representation map is canonical.

In this letter we construct a hierarchy of kinetic equations which produces the basic hierarchy of discrete Lax equations. The latter has the form [1, 2]:

$$L_{,t} = [(L^m)_+, L] \qquad m \in \mathbb{Z}_+ \tag{1}$$

$$L = \zeta + \sum_{i=0}^{\infty} a_i \zeta^{-i} \tag{2}$$

where the a_i can be thought of as functions of either $n \in \mathbb{Z}$ or $x \in \mathbb{R}$; in either case, the commutation rule of ζ and the a is

$$\zeta^l a = \Delta^l(a)\zeta^l \qquad l \in \mathbb{Z} \tag{3}$$

where $[\Delta^l(a)](n) := a(n+l)$ and $[\Delta^l(a)](x) := a(x + l\varepsilon)$, ε being a (small) formal parameter. Thus, Δ acts on functions as dual to the argument shift, and ζ is the operator version of Δ. The additional notation employed in (1) is

$$(\sum \varphi_l \zeta^l)_+ := \sum_{l \geq 0} \varphi_l \zeta^l \tag{4}$$

$$[X, Y] := XY - YX. \tag{5}$$

For the first non-trivial case $m = 1$, (1) becomes

$$a_{i,t} = (\Delta - 1)(a_{i+1}) + a_i(1 - \Delta^{-i})(a_0) \tag{6}$$

which turns, in the continuous limit $\varepsilon \to 0$, $\Delta = \exp(\varepsilon \partial / \partial x)$, into

$$a_{i,t} = a_{i+1,x} + \mathrm{i} a_i a_{0,x} \qquad i \in \mathbb{Z}_+. \tag{7}$$

Now, the *infinite-component* system (7) results, as is easy to verify, from the following dynamics for the *one-particle* distribution function $f = f(x, p, t)$:

$$f_{,t} = pf_{,x} - a_{0,x}(pf)_{,p} \tag{8}$$

$$a_n = \int_{-\infty}^{\infty} fp^n \, \mathrm{d}p. \tag{9}$$

A similar kinetic representation can be given to the full dispersive system (6) and even to the whole hierarchy (1), as follows.

Set

$$(\sum \varphi_l \zeta^l)_{\leq 0} := \sum_{l \leq 0} \varphi_l \zeta^l$$

$$\mathrm{Smbl}(\sum \varphi_l \zeta^l) := \sum \varphi_l p^l.$$

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Property 1. For any $Q = \Sigma\, q_l \zeta^l$, the system of motion equations

$$L_{,t} = [Q_+, L]_{\leq 0} \tag{10}$$

results, via the moments map (9), from the single kinetic equation

$$f_{,t} = [\mathrm{Smbl}(Q_+), f]_{(1)} \tag{11}$$

where

$$[X, Y]_{(1)} := \sum_{r\geq 0} \frac{\varepsilon^r}{r!} [(p\partial_p)^r(X)\partial^r(Y) - \partial^r(X)(\partial_p p)^r(Y)] \qquad \partial := \partial/\partial x. \tag{12}$$

Remark. $[X, Y]_{(1)} \neq -[Y, X]_{(1)}$.

Let $B = (B_{ij})$ be the (first) Hamiltonian structure [1, 2] of the hierarchy (1):

$$B_{ij} = \Delta^j a_{i+j} - a_{i+j}\Delta^{-i}. \tag{13}$$

Thus, the motion equations with a Hamiltonian $H = H(\boldsymbol{a})$ are

$$a_{i,t} = \sum (\Delta^j a_{i+j} - a_{i+j}\Delta^{-i})(H_j) \qquad H_j := \delta H / \delta a_j. \tag{14}$$

Property 2. For any Hamiltonian $H = H(f)$, consider the motion equation

$$f_{,t} = \bar{B}\left(\frac{\delta H}{\delta f}\right) \tag{15}$$

$$\bar{B} := \sum \frac{\varepsilon^r}{r!} [\partial^r f (p\partial_p)^r - (\partial_p p)^r f \partial^r]. \tag{16}$$

Then the matrix $\bar{B}$ (16) is Hamiltonian.

Property 3. If H in (15) depends upon f only through the a, the Hamiltonian system (15) implies the Hamiltonian system (14). In other words, the moments map (9) is a canonical (=Hamiltonian) map between the Hamiltonian structures (16) and (13).

Corollary 1. For $Q = L^m$, $m \in \mathbb{Z}_+$, the kinetic representation (11) is the same as (15) with $H = \mathrm{Res}(L^{m+1})/(m+1)$, where

$$\mathrm{Res}\left(\sum \varphi_l \zeta^l\right) := \varphi_0. \tag{17}$$

Indeed, the moments map (9) is injective, and the system (1) can be put into the Hamiltonian form (14) with the Hamiltonian $H = \mathrm{Res}(L^{m+1})/(m+1)$ [1, 2].

Corollary 2. When H runs through the set $\{\mathrm{Res}(L^{m+1})/(m+1) | m \in \mathbb{Z}_+\}$, the kinetic flows (15) all commute.

Indeed, this set of Hamiltonians is in involution with respect to the Hamiltonian structure B (13) since the Hamiltonian flows (1) all commute [1, 2]; by property 3, the moments map (9) is canonical; hence, these Hamiltonians are in involution in the Hamiltonian structure (16); therefore, the kinetic flows (15) commute as well.

Proof of 1. Equations (10) and (11) are, in full,

$$a_{i,t} = \sum [\Delta^j(a_{i+j})q_j - a_{i+j}\Delta^{-i-j}(q_j)] \tag{18}$$

$$f_{,t} = \sum p^j \Delta^j(f) q_j - \sum \frac{\varepsilon^r}{r!} p^j (\partial_p p)^r(f) \partial^r(q_j). \tag{19}$$

Applying to (19) the operation $\int_{-\infty}^{\infty} p^i \, \mathrm{d}p$ and using the formula

$$\int_{-\infty}^{\infty} p^i (\partial_p p)^r (f p^j) \, \mathrm{d}p = (-\mathrm{i})^r a_{i+j} \tag{20}$$

we obtain (18).

Proof of 2. The multiplication of symbols of pseudodifferential operators is an associative operation given by the formula [3]

$$\varphi \circ \psi = \sum_r \frac{1}{r!} \partial_\xi^r (\varphi) \partial^r (\psi)$$

for functions of x and ξ. Changing ξ into $p = \exp(-\varepsilon\xi)$ results in the multiplication

$$\varphi \circ \psi = \sum \frac{(-\varepsilon)^r}{r!} (p\partial_p)^r (\varphi) \partial^r (\psi)$$

which produces the Lie algebra structure

$$[\varphi, \psi]_{(2)} := \psi \circ \varphi - \varphi \circ \psi = \sum \frac{(-\varepsilon)^r}{r!} [\partial^r(\varphi)(p\partial_p)^r(\psi) - (p\partial_p)^r(\varphi)\partial^r(\psi)]. \tag{21}$$

Denoting by f the coordinate on the dual space to the Lie algebra g with the commutator (21), we compute the associated Hamiltonian matrix $\bar{B} := B(\mathrm{g})$ by the standard rule [2]:

$$\varphi \bar{B}(\psi) \sim f[\varphi, \psi]_{(2)}$$

which yields (16).

Proof of 3. If H depends upon f only through the a then

$$\frac{\delta H}{\delta f} = \sum p^k H_k$$

and (15) becomes

$$\begin{aligned} f_{,t} &= \sum \frac{\varepsilon^r}{r!} [\partial^r f (p\partial_p)^r - (\partial_p p)^r f \partial^r)](p^k H_k) \\ &= \sum \frac{\varepsilon^r}{r!} [\partial^r (f H_k) k^r p^k - (\partial_p p)^r (f p^k) \partial^r (H_k)] \\ &= \sum \Delta^k (f H_k) p^k - \sum \frac{\varepsilon^r}{r!} (\partial_p p)^r (f p^k) \partial^r (H_k). \end{aligned} \tag{22}$$

Applying to (22) the operation $\int_{-\infty}^{\infty} p^i \, \mathrm{d}p$ and using formula (20) we get (14).

Taking the continuous limit of the properties 1-3 yields the corresponding properties of the continuous zero-dispersion hierarchy commuting with (7).

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References

[1] Kupershmidt B A 1982 *Lett. Math. Phys.* **6** 85
[2] Kupershmidt B A 1985 *Discrete Lax Equations and Differential-Difference Equations* (Paris: Astérisque)
[3] Manin Yu I 1979 *J. Sov. Math.* **11** 1